Induced Lorentz- and CPT-violating Chern-Simons term in QED: Fock-Schwinger proper time method

J. -M. Chung[a*] and B. K. Chung[a,b†]
[a] *Asia Pacific Center for Theoretical Physics, Seoul 130-012, Korea*
[b] *Research Institute for Basic Sciences and Department of Physics*
Kyung Hee University, Seoul 130-701, Korea

Abstract

Using the Fock-Schwinger proper time method, we calculate the induced Chern-Simons term arising from the Lorentz- and CPT-violating sector of quantum electrodynamics with a $b_\mu \bar{\psi}\gamma^\mu\gamma_5\psi$ term. Our result to all orders in b coincides with a recent linear-in-b calculation by Chaichian et al. [hep-th/0010129 v2]. The coincidence was pointed out by Chung [Phys. Lett. **B461** (1999) 138] and Pérez-Victoria [Phys. Rev. Lett. **83** (1999) 2518] in the standard Feynman diagram calculation with the nonperturbative-in-b propagator.

PACS number(s): 12.20.-m; 11.30.Cp

Typeset using REVTEX

*Electronic address: jmchung@apctp.org

†Electronic address: bkchung@khu.ac.kr

The Lorentz- and CPT-violating Chern-Simons modification to the Maxwell theory was first proposed a decade ago [1]. An important feature of the Chern-Simons term [2] is that Lagrange density is not gauge invariant, but the action and equations of motion are gauge invariant. In their Lorentz violating extension of the standard model, Colladay and Kostelecký posed the question whether such a term is induced when the Lorentz- and CPT-violating term $\bar{\psi}\not{b}\gamma_5\psi$ (b_μ is a constant four vector) is added to the conventional Lagrangian of QED [3].

Recently, several calculations have been carried out to determine the radiatively induced Chern-Simons term from the Lorentz- and CPT-violating fermion sector [3–15]:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - e\not{A} - \gamma_5\not{b} - m)\psi \ . \tag{1}$$

Jackiw and Kostelecký clarified that the induced Chern-Simons term is not uniquely determined [8]. It depends on whether one uses a nonperturbative definition or a perturbative definition of the theory defined by Eq. (1);[1] In a nonperturbative formalism, radiative corrections induce a definite nonzero Chern-Simons term, while when a perturbative formalism is used, radiative corrections are finite but undetermined. The regularization scheme one chooses to adopt can generate further ambiguity in both nonperturbative and perturbative formalisms.

In the standard Feynman diagram calculation with the nonperturbative-in-b propagator, Chung [6] and Pérez-Victoria [7] demonstrated that the result to all orders in b coincides with the previous linear-in-b calculation by Chung and Oh [4] as well as Jackiw and Kostelecký [8]. More recently, nonstandard approaches were employed also in the calculation of the induced Chern-Simons term. Chan [10] and Chaichian et al. [12], used the covariant derivative expansion [17] and the Fock-Schwinger proper time method [18,19], respectively, to obtain the induced Chern-Simons term. The common feature in these two nonstandard approaches is to develop a series of local effective Lagrangian in powers of $\Pi_\mu = i\partial_\mu - eA_\mu$, rather than in powers of $i\partial_\mu$ and A_μ separately.

The purpose of this work is to calculate, using the Fock-Schwinger proper time method, the induced Chern-Simons term arising from the Lorentz- and CPT-violating sector of QED with a $\bar{\psi}\not{b}\gamma_5\psi$ term keeping the full b dependence in order to see whether the coincidence of all-order-in-b result with linear-in-b calculation takes place in this nonstandard approach.

The effective action, Γ_{eff}, of the theory defined by Eq. (1) is given by

$$\Gamma_{\text{eff}} = -i\ln\left(\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \mathrm{e}^{i\int d^4x\mathcal{L}}\right) = -i\mathrm{Tr}\ln(i\not{\partial} - e\not{A} - \gamma_5\not{b} - m) \ , \tag{2}$$

where the trace "Tr" is taken over both spinor indices (of any combinations of 4×4 Dirac matrices) and space-time coordinates.

Let us decompose the trace in Eq. (2) in the following manner, as was done in Eq. (3) of Ref. [12]:

---

[1] In the nonperturbative definition, we use the b-exact propagator $S(p) = \frac{i}{\not{p}-m-\gamma_5\not{b}}$ for the standard perturbation calculation. (See the work of Colladay and Kostelecký [16] for detailed analysis of the extended Dirac theory with this propagator.) In the perturbative definition, the propagator has the b-independent form $S(p) = \frac{i}{\not{p}-m}$ and $-\gamma_5\not{b}$ is considered as an interaction vertex.

$$\mathrm{Tr}\ln(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_5 b\!\!\!/ - m) = \mathrm{Tr}\ln(i\partial\!\!\!/ - eA\!\!\!/ - m) + \int_0^1 dz\,\mathrm{Tr}\Big(\frac{1}{-i\partial\!\!\!/ + eA\!\!\!/ + z\gamma_5 b\!\!\!/ + m}\gamma_5 b\!\!\!/\Big)\ . \tag{3}$$

Then, the effective action can be written down as follows:

$$\Gamma_{\text{eff}} = \Gamma_{\text{eff}}^{(0)} + \Gamma_{\text{eff}}^{(1)}\ ,$$

where

$$\begin{aligned}\Gamma_{\text{eff}}^{(0)} &= -i\mathrm{Tr}\ln(i\partial\!\!\!/ - eA\!\!\!/ - m)\ ,\\ \Gamma_{\text{eff}}^{(1)} &= -i\int_0^1 dz\,\mathrm{Tr}\Big(\frac{1}{-i\partial\!\!\!/ + eA\!\!\!/ + z\gamma_5 b\!\!\!/ + m}\gamma_5 b\!\!\!/\Big)\ .\end{aligned} \tag{4}$$

The induced Chern-Simons term is contained in $\Gamma_{\text{eff}}^{(1)}$. By neglecting the dependence of b in the denominator of the trace in $\Gamma_{\text{eff}}^{(1)}$, the authors of Ref. [12] obtained the undetermined induced Chern-Simons action[2]

$$\Gamma_{\text{CS}}^{(1)} = \frac{ce^2}{4\pi^2}\int d^4x\,\epsilon^{\mu\nu\lambda\rho}b_\mu A_\nu F_{\lambda\rho}\ . \quad (c \text{ being an arbitrary contant})$$

If one keeps the full dependence of b in the calculation of $\Gamma_{\text{eff}}^{(1)}$ for extracting the induced Chern-Simons term, one expects that the induced Chern-Simons action would take the following form:

$$\Gamma_{\text{CS}}^{(1)} = \frac{ce^2}{4\pi^2}\Big[1 + f\Big(\frac{b^2}{m^2}\Big)\Big]\int d^4x\,\epsilon^{\mu\nu\lambda\rho}b_\mu A_\nu F_{\lambda\rho}\ , \tag{5}$$

where $f(b^2/m^2)$ is some function of a single argument b^2/m^2. In what follows, we are to show by explicit calculation that this function $f(b^2/m^2)$, indeed, vanishes.

Now, let us introduce a (fermionic) Green's function $G(x,x')$ as the inverse of the operator $-i\partial\!\!\!/ + eA\!\!\!/(x) + z\gamma_5 b\!\!\!/ + m$, i.e., let us assume that $G(x,x')$ satisfies the following inhomogeneous differential equation

$$[-i\partial\!\!\!/ + eA\!\!\!/(x) + z\gamma_5 b\!\!\!/ + m]G(x,x') = \delta^4(x-x')\ . \tag{6}$$

Then, $\Gamma_{\text{eff}}^{(1)}$ of Eq. (4) can be written as

$$\Gamma_{\text{eff}}^{(1)} = \int d^4x\mathcal{L}_{\text{eff}}^{(1)} = -i\int d^4x\int_0^1 dz\,\mathrm{tr}[G(x,x')\gamma_5 b\!\!\!/]_{x'\to x}\ . \tag{7}$$

The trace "tr" in this equation is taken over the spinor indices only and the limit $x' \to x$ is taken by averaging the forms obtained by letting x' approach from the future and from the past [19]. Further, if one introduces a (bosonic) Green's function Δ as follows:

---

[2] This undeterminicity arises from an intrinsic ambiguity in the limit $\lim_{x\to 0} x_\mu x_\nu/x^2$. (See Eq. (36) below.) This limit has directional dependence in a strict mathematical sense as emphasized in Ref. [12].

$$G(x,x') = [i\partial\!\!\!/ - eA\!\!\!/(x) - z\gamma_5 b\!\!\!/ + m]\Delta(x,x') \ , \tag{8}$$

or, equivalently,

$$G(x,x') = [-i\partial'_\mu - eA_\mu(x')]\Delta(x,x')\gamma^\mu + \Delta(x,x')[-z\gamma_5 b\!\!\!/ + m] \ , \tag{9}$$

then, one finds that Eq. (6) becomes

$$\mathcal{H}\Delta(x,x') = \delta^4(x-x') \ , \tag{10}$$

where $\mathcal{H}$ is defined as

$$\mathcal{H} = -\Pi^2 + m^2 + z^2 b^2 + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu} + 2iz\sigma^{\mu\nu}\Pi_\mu b_\nu \gamma_5 \ , \tag{11}$$

with $\Pi_\mu \equiv i\partial_\mu - eA_\mu$. Borrowing the quantum mechanical matrix element notation, the Green's function $\Delta(x',x'')$ can be expressed as follows:

$$\Delta(x',x'') = \langle x'|\mathcal{H}^{-1}|x''\rangle = i\int_0^\infty ds\, \langle x'|e^{-i\mathcal{H}s}|x''\rangle \ . \tag{12}$$

In the s integration, a convergence factor $-i\epsilon$ ($\epsilon > 0$) in $\mathcal{H}$ is understood. The idea of the Fock-Schwinger proper time method is to consider $\mathcal{H}$ in Eqs. (10) and (11) as a Hamiltonian that governs the evolution in "time" s of a hypothetical quantum mechanical system. Then the integrand in Eq. (12) becomes a transition amplitude $\langle x'(s)|x''(0)\rangle$. Thus, the solution $\Delta(x',x'')$ can be written, in terms of this transition amplitude, as follows:

$$\Delta(x',x'') = i\int_0^\infty ds\, \langle x'(s)|x''(0)\rangle \ . \tag{13}$$

Now, let us evaluate the transition amplitude $\langle x'(s)|x''(0)\rangle$. It satisfies the evolution equation

$$i\partial_s \langle x'(s)|x''(0)\rangle = \langle x'(s)|\mathcal{H}\big(x(s),\Pi(s)\big)|x''(0)\rangle \ , \tag{14}$$

with the boundary condition

$$\langle x'(s)|x''(0)\rangle_{s\to 0} = \delta^4(x'-x'') \ . \tag{15}$$

In order to obtain the Hamiltonian in the Heisenberg representation, $\mathcal{H}\big(x(s),\Pi(s)\big)$, one first has to know the evolutions of x and Π. Using the commutation relations

$$[x^\mu, \Pi^\nu] = -ig^{\mu\nu} \ , \quad [\Pi^\mu, \Pi^\nu] = -ieF^{\mu\nu} \ ,$$

one can obtain the equations of motion for the operators $x(s)$ and $\Pi(s)$:

$$\begin{aligned} \dot{x}_\mu &= i[\mathcal{H}, x_\mu] = 2\Pi_\mu - 2iz\sigma_{\mu\nu}b^\nu\gamma_5 \ , \\ \dot{\Pi}_\mu &= i[\mathcal{H}, \Pi_\mu] = 2eF_{\mu\nu}\Pi^\nu + ie\partial^\nu F_{\mu\nu} + \frac{e}{2}\partial_\mu F_{\rho\nu}\sigma^{\rho\nu} - 2izeF_{\mu\rho}\sigma^{\rho\nu}b_\nu\gamma_5 \ . \end{aligned} \tag{16}$$

In the constant $F_{\mu\nu}$ approximation, these equations are solved, in matrix notation, as follows:

$$\begin{aligned}\Pi(s) &= \mathrm{e}^{2eFs}(\Pi(0) - iz\sigma b\gamma_5) + iz\sigma b\gamma_5\ ,\\ x(s) &= x(0) + (eF)^{-1}\Big(\mathrm{e}^{2eFs} - 1\Big)(\Pi(0) - iz\sigma b\gamma_5)\ .\end{aligned} \tag{17}$$

From these two relations, one can find

$$\begin{aligned}\Pi(s) &= \frac{1}{2}eF\mathrm{e}^{eFs}[\sinh(eFs)]^{-1}[x(s) - x(0)] + iz\sigma b\gamma_5\ ,\\ \Pi(0) &= \frac{1}{2}eF\mathrm{e}^{-eFs}[\sinh(eFs)]^{-1}[x(s) - x(0)] + iz\sigma b\gamma_5\ ,\\ \Pi^2(s) &= [x(s) - x(0)]K[x(s) - x(0)]\\ &\quad + ize[x(s) - x(0)]\mathrm{e}^{-eFs}[\sinh(eFs)]^{-1}F\sigma b\gamma_5 - 3z^2b^2\ ,\end{aligned} \tag{18}$$

where

$$K = \frac{1}{4}e^2F^2[\sinh(eFs)]^{-2}\ .$$

Further, using the commutation relation

$$[x_\mu(s), x_\nu(0)] = i\Big[(eF)^{-1}(\mathrm{e}^{2eFs} - 1)\Big]_{\mu\nu}\ ,$$

one can find[3]

$$x(0)Kx(s) = x(s)Kx(0) - \frac{i}{2}\mathrm{TR}[eF\coth(eFs)]\ . \tag{19}$$

From Eqs. (11), (18), and (19), one finally obtains

$$\begin{aligned}\mathcal{H}(x(s), \Pi(s)) &= -x(s)Kx(s) + 2x(s)Kx(0) - x(0)Kx(0)\\ &\quad -\frac{i}{2}\mathrm{TR}[eF\coth(eFs)] - \frac{e}{2}\mathrm{TR}(\sigma F) + m^2 - 2z^2b^2\ .\end{aligned}$$

Therefore, Eq. (14) becomes

$$\begin{aligned}i\partial_s\langle x'(s)|x''(0)\rangle &= \Big[-(x' - x'')K(x' - x'') - \frac{i}{2}\mathrm{TR}[eF\coth(eFs)]\\ &\quad -\frac{e}{2}\mathrm{TR}(\sigma F) + m^2 - 2z^2b^2\Big]\langle x'(s)|x''(0)\rangle\ ,\end{aligned}$$

whose solution is

$$\begin{aligned}\langle x'(s)|x''(0)\rangle &= C(x', x'')s^{-2}\mathrm{e}^{-L(s)}\exp\Big[-\frac{i}{4}(x' - x'')eF\coth(eFs)(x' - x'')\\ &\quad + \frac{ie}{2}\mathrm{TR}(\sigma F)s - i(m^2 - 2z^2b^2)s\Big]\ ,\end{aligned} \tag{20}$$

[3]The trace denoted by "TR" extends over Lorentz indices only: $\mathrm{TR}(AB) = A_{\mu\nu}B^{\nu\mu}$. In particular, note that the expression $\mathrm{TR}(\sigma F) = \sigma_{\mu\nu}F^{\nu\mu}$ is still a matrix valued quantity because each $\sigma_{\mu\nu}$ is a 4×4 matrix.

where

$$L(s) = \frac{1}{2}\mathrm{TR}\ln[(eFs)^{-1}\sinh(eFs)] \ .$$

From the definition of the operator Π and the first two relations in Eq. (18), one can find

$$\begin{aligned}
&[i\partial' - eA(x')]\langle x'(s)|x''(0)\rangle = \Big(\frac{1}{2}eF[\coth(eFs)+1](x'-x'') + iz\sigma b\gamma_5\Big)\langle x'(s)|x''(0)\rangle \ , \\
&[-i\partial'' - eA(x'')]\langle x'(s)|x''(0)\rangle = \Big(\frac{1}{2}eF[\coth(eFs)-1](x'-x'') + iz\sigma b\gamma_5\Big)\langle x'(s)|x''(0)\rangle \ , \quad (21)
\end{aligned}$$

from which, in conjunction with Eq. (20), the differential equations for $C(x', x'')$ are obtained:

$$\begin{aligned}
\Big[i\partial' - eA(x') - \frac{e}{2}F(x'-x'') - iz\sigma b\gamma_5\Big]C(x',x'') = 0 \ , \\
\Big[-i\partial'' - eA(x'') + \frac{e}{2}F(x'-x'') - iz\sigma b\gamma_5\Big]C(x',x'') = 0 \ .
\end{aligned}$$

Two forms of the solution for $C(x', x'')$ are obtained:

$$C(x',x'') = C_1(x'')\exp\Big[z(x'-x'')\sigma b\gamma_5 - ie\int_{x''}^{x'} dx^\mu\Big(A(x) + \frac{1}{2}F(x-x'')\Big)_\mu\Big] \ , \quad (22)$$

$$C(x',x'') = C_2(x')\exp\Big[z(x'-x'')\sigma b\gamma_5 + ie\int_{x'}^{x''} dx^\mu\Big(A(x) + \frac{1}{2}F(x-x')\Big)_\mu\Big] \ , \quad (23)$$

Since both integrals in Eqs. (22) and (23) are independent of the integration path due to the Stokes' theorem, one may choose the integration path to be a straight line connecting x' and x'', i.e., $x = x'' + t(x' - x'')$. Then, it is readily shown that

$$\begin{aligned}
\int_{x''}^{x'} dx^\mu\Big(F(x-x'')\Big)_\mu &= \int_0^1 dt(x'-x'')F(x'-x'') = 0 \ , \\
\int_{x'}^{x''} dx^\mu\Big(F(x-x')\Big)_\mu &= \int_1^0 dt(x'-x'')F(x'-x'')(1-t) = 0 \ .
\end{aligned}$$

With these vanishing integrals in mind, the comparison of two expressions for $C(x', x'')$ in Eqs. (22) and (23) leads us to the following conclusion: $C_1(x'') = C_2(x') = \mathcal{C}$ (constant). The constant $\mathcal{C}$ is finally determined by Eq. (15) as

$$\mathcal{C} = -\frac{i}{(4\pi)^2} \ .$$

Therefore, the complete form of the transition amplitude is given as

$$\begin{aligned}
\langle x'(s)|x''(0)\rangle = -\frac{i}{(4\pi)^2}\Phi(x',x'')s^{-2}\mathrm{e}^{-L(s)}\exp\Big[&-\frac{i}{4}(x'-x'')eF\coth(eFs)(x'-x'') \\
&+\frac{ie}{2}\mathrm{TR}(\sigma F)s - i(m^2 - 2z^2b^2)s\Big] \ , \quad (24)
\end{aligned}$$

where

$$\Phi(x',x'') = \exp\left[z(x'-x'')\sigma b\gamma_5 - ie\int_{x''}^{x'} dx^\mu A_\mu(x)\right] . \tag{25}$$

Now, let us compute $\mathrm{tr}[G(x,x')\gamma_5 \not{b}]$ in Eq. (7). From Eqs. (8), (9), and (13), we obtain

$$\begin{aligned}
\mathrm{tr}[G(x,x')\gamma_5\not{b}] &= \mathrm{tr}\left[(i\gamma_5 b^\mu - \gamma_5\sigma^{\mu\nu}b_\nu)\int_0^\infty ds\langle x(s)|\Pi_\mu(s)|x'(0)\rangle\right] \\
&\quad + izb^2\mathrm{tr}\left[\int_0^\infty ds\langle x(s)|x'(0)\rangle\right] \\
&= \mathrm{tr}\left[(-i\gamma_5 b^\mu - \gamma_5\sigma^{\mu\nu}b_\nu)\int_0^\infty ds\langle x(s)|\Pi_\mu(0)|x'(0)\rangle\right] \\
&\quad + izb^2\mathrm{tr}\left[\int_0^\infty ds\langle x(s)|x'(0)\rangle\right] .
\end{aligned}$$

Averaging these two equivalent expressions gives

$$\begin{aligned}
\mathrm{tr}[G(x,x')\gamma_5\not{b}] &= \frac{ib^\mu}{2}\mathrm{tr}\left[\gamma_5\int_0^\infty ds\langle x(s)|[\Pi_\mu(s)-\Pi_\mu(0)]|x'(0)\rangle\right] \\
&\quad -\frac{b_\nu}{2}\mathrm{tr}\left[\gamma_5\sigma^{\mu\nu}\int_0^\infty ds\langle x(s)|[\Pi_\mu(s)+\Pi_\mu(0)]|x'(0)\rangle\right] \\
&\quad + izb^2\mathrm{tr}\left[\int_0^\infty ds\langle x(s)|x'(0)\rangle\right] .
\end{aligned} \tag{26}$$

From the first two relations in Eq. (18), we have

$$\begin{aligned}
&\langle x(s)|\Pi_\mu(s)-\Pi_\mu(0)|x'(0)\rangle = [eF(x-x')]_\mu\langle x(s)|x'(0)\rangle , \\
&\langle x(s)|\Pi_\mu(s)+\Pi_\mu(0)|x'(0)\rangle = [eF\coth(eFs)(x-x') + iz\sigma b\gamma_5]_\mu\langle x(s)|x'(0)\rangle .
\end{aligned} \tag{27}$$

Using Eqs. (24) and (27), we finally rewrite Eq. (26) as follows:

$$\begin{aligned}
\mathrm{tr}[G(x,x')\gamma_5\not{b}] &= \frac{1}{(4\pi)^2}\Phi_0(x,x')\int_0^\infty \frac{ds}{s^2}\,\mathrm{e}^{-i(m^2-2z^2b^2)s}\,\mathrm{e}^{-L(s)}\,\mathrm{e}^{-\frac{i}{4}(x-x')eF\coth(eFs)(x-x')} \\
&\quad \times\Big\{\frac{1}{2}eF_{\mu\nu}(x-x')^\nu b^\mu\mathrm{tr}\left[\gamma_5\,\mathrm{e}^{z(x-x')\sigma b}\,\mathrm{e}^{\frac{ie}{2}\mathrm{TR}(\sigma F)s}\right] \\
&\quad +\frac{ie}{2}F_{\mu\alpha}[\coth(eFs)]^{\alpha\beta}(x-x')_\beta b_\nu\mathrm{tr}\left[\gamma_5\sigma^{\mu\nu}\,\mathrm{e}^{z(x-x')\sigma b}\,\mathrm{e}^{\frac{ie}{2}\mathrm{TR}(\sigma F)s}\right] \\
&\quad -\frac{z}{2}b^2\mathrm{tr}\left[\mathrm{e}^{z(x-x')\sigma b}\,\mathrm{e}^{\frac{ie}{2}\mathrm{TR}(\sigma F)s}\right]\Big\} ,
\end{aligned} \tag{28}$$

where $\Phi_0(x,x')$ is the function $\Phi(x,x')$ in Eq. (25) at $z=0$.

The vacuum current vector $\langle J_\mu(x)\rangle$ associated with b is obtained from $\Gamma^{(1)}_{\text{eff}}$ by variation of $A^\mu(x)$:

$$\langle J_\mu(x)\rangle = \frac{\delta\Gamma^{(1)}_{\text{eff}}}{\delta A^\mu(x)} . \tag{29}$$

Thus, from Eqs. (7) and (28), we have, after deforming the integration path of is to the positive real axis,

$$
\begin{aligned}
\langle J_\mu(x)\rangle = {} & \frac{ie}{(4\pi)^2}(x-x')_\mu \Phi_0(x,x') \int_0^1 dz \int_0^\infty \frac{ds}{s^2} \\
& \times \mathrm{e}^{-(m^2-2z^2b^2)s}\ \mathrm{e}^{-\ell(s)}\ \mathrm{e}^{\frac{1}{4}(x-x')eF\cot(eFs)(x-x')} \\
& \times \Big\{ -\frac{1}{2} eF_{\rho\nu}(x-x')^\nu b^\rho \mathrm{tr}\Big[\gamma_5\, \mathrm{e}^{z(x-x')\sigma b\gamma_5}\ \mathrm{e}^{\frac{e}{2}\mathrm{TR}(\sigma F)s}\Big] \\
& + \frac{e}{2} F_{\rho\alpha}[\cot(eFs)]^{\alpha\beta}(x-x')_\beta b_\nu \mathrm{tr}\Big[\gamma_5\sigma^{\rho\nu}\, \mathrm{e}^{z(x-x')\sigma b\gamma_5}\ \mathrm{e}^{\frac{e}{2}\mathrm{TR}(\sigma F)s}\Big] \\
& + \frac{z}{2} b^2 \mathrm{tr}\Big[\mathrm{e}^{z(x-x')\sigma b\gamma_5}\ \mathrm{e}^{\frac{e}{2}\mathrm{TR}(\sigma F)s}\Big]\Big\}_{x'\to x} ,
\end{aligned}
\tag{30}
$$

where

$$
\ell(s) = \frac{1}{2}\mathrm{TR}\ln[(eFs)^{-1}\sin(eFs)] .
$$

Using the eigenvalue technique [19], $\mathrm{e}^{-\ell(s)}$ is determined as

$$
\mathrm{e}^{-\ell(s)} = \frac{(es)^2\mathcal{G}}{\mathrm{Im}\ \cosh(esX)} ,
\tag{31}
$$

where

$$
X = \sqrt{2(\mathcal{F}+i\mathcal{G})} , \quad \mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\mathbf{E}^2-\mathbf{B}^2) , \quad \mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathbf{E}\cdot\mathbf{B}
$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. From the matrix decomposition formulas

$$
\begin{aligned}
\mathrm{e}^{z(x-x')\sigma b\gamma_5} &= a_1 + a_2(x-x')\sigma b\gamma_5 , \\
\mathrm{e}^{\frac{e}{2}\mathrm{TR}(\sigma F)s} &= c_1 + c_2\mathrm{TR}(\sigma F) + ic_3\gamma_5 + ic_4\mathrm{TR}(\sigma F)\gamma_5 ,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \sum_{N=0}^{\infty} \frac{z^{2N}}{(2N)!}[(x-x')^2b^2 - ((x-x')\cdot b)^2]^N , \\
a_2 &= \sum_{N=0}^{\infty} \frac{z^{2N+1}}{(2N+1)!}[(x-x')^2b^2 - ((x-x')\cdot b)^2]^N , \\
c_1 &= \mathrm{Re}\ \cosh(esX) , \qquad c_2 = \mathrm{Re}\,[\sinh(esX)/(2X)] , \\
c_3 &= \mathrm{Im}\ \cosh(esX) , \qquad c_4 = \mathrm{Im}\,[\sinh(esX)/(2X)] ,
\end{aligned}
$$

the trace quantities in the integrand of Eq. (30) are evaluated as follows:

$$
\begin{aligned}
\mathrm{tr}\Big[\mathrm{e}^{z(x-x')\sigma b\gamma_5}\ \mathrm{e}^{\frac{e}{2}\mathrm{TR}(\sigma F)s}\Big] &= 4a_1c_1 - 8ia_2c_2(x-x')\tilde{F}b - 8ia_2c_4(x-x')Fb , \\
\mathrm{tr}\Big[\gamma_5\, \mathrm{e}^{z(x-x')\sigma b\gamma_5}\ \mathrm{e}^{\frac{e}{2}\mathrm{TR}(\sigma F)s}\Big] &= 4ia_1c_3 - 8a_2c_2(x-x')Fb + 8a_2c_4(x-x')\tilde{F}b , \\
\mathrm{tr}\Big[\gamma_5\sigma^{\rho\nu}\, \mathrm{e}^{z(x-x')\sigma b\gamma_5}\ \mathrm{e}^{\frac{e}{2}\mathrm{TR}(\sigma F)s}\Big] &= -8ia_1\Big(c_2\tilde{F}^{\rho\nu} + c_4F^{\rho\nu}\Big) \\
& + 4a_2\Big(c_1[(x-x')^\rho b^\nu - (x-x')^\nu b^\rho] + ic_3(x-x')_\alpha b_\beta\epsilon^{\alpha\beta\rho\nu}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 8ia_2\Big(c_2[b^\rho\big((x-x')F\big)^\nu - b^\nu\big((x-x')F\big)^\rho \\
&+ (x-x')^\nu(bF)^\rho - (x-x')^\rho(bF)^\nu] \\
&+ 4ic_4[\epsilon^{\rho\nu\alpha\beta}(x-x')_\alpha(bF)_\beta - \epsilon^{\rho\nu\alpha\beta}b_\alpha\big((x-x')F\big)_\beta \\
&- b^\rho\big((x-x')\tilde{F}\big)^\nu + b^\nu\big((x-x')\tilde{F}\big)^\rho \\
&+ (x-x')^\rho(b\tilde{F})^\nu - (x-x')^\nu(b\tilde{F})^\rho]\Big)\,. \qquad (32)
\end{aligned}
$$

Now substituting Eqs. (31) and (32) into Eq. (30) and discarding terms that do not contribute to the Chern-simons current, we obtain

$$
\begin{aligned}
\langle J_\mu(x)\rangle = {} & \frac{e^3\mathcal{G}}{4\pi^2}(x-x')_\mu\Phi_0(x,x')\int_0^1 dz\int_0^\infty ds\,\frac{1}{\mathrm{Im}\,\cosh(esX)} \\
& \times \mathrm{e}^{-(m^2-2z^2b^2)s}\,\mathrm{e}^{\frac{1}{4}(x-x')eF\coth(eFs)(x-x')} \\
& \times\Big\{a_1c_2eF_{\rho\alpha}[\cot(eFs)]^{\alpha\beta}(x-x')_\beta b_\nu\tilde{F}^{\rho\nu} + a_2c_2zb^2(x-x')_\rho\tilde{F}^{\rho\nu}b_\nu\Big\}_{x'\to x}\,. \qquad (33)
\end{aligned}
$$

Thus, the current $\langle J_\mu(x)\rangle$ of Eq. (33) is cast, in the leading order approximation in the coupling constant e, to the following form:[4]

$$
\begin{aligned}
\langle J_\mu(x)\rangle = {} & \frac{e^2}{8\pi^2}\int_0^1 dz\Big[(x-x')_\mu(x-x')_\rho\Phi_0(x,x')\tilde{F}^{\rho\nu}b_\nu \\
& \times\int_0^\infty ds\Big(\frac{a_1}{s^2}+\frac{a_2zb^2}{s}\Big)\mathrm{e}^{-(m^2-2z^2b^2)s+(x-x')^2/(4s)}\Big]_{x'\to x}\,. \qquad (34)
\end{aligned}
$$

In the light of b-perturbation theory, the insertions of $\not{b}\gamma_5$ are revealed best via the expansion of the factor $\mathrm{e}^{2z^2b^2s}$ [in Eq. (34)] in powers of its exponent. Then using an integration formula

$$
\int_0^\infty ds s^{n-2}\,\mathrm{e}^{-sy-1/s} = 2y^{(1-n)/2}K_{n-1}\big(2y^{1/2}\big)\,,
$$

with $K_n(x)$ being the modified Bessel function of order n, the Chern-Simons current $\langle J_\mu(x)\rangle$ in Eq. (34) is evaluated as follows:

$$
\begin{aligned}
\langle J_\mu(x)\rangle = {} & \frac{e^2}{2\pi^2}\int_0^1 dz\Big[-\Phi_0(x,x')\frac{(x-x')_\mu(x-x')_\rho}{(x-x')^2}\,b_\nu\tilde{F}^{\rho\nu} \\
& \times\sum_{N=0}^\infty\frac{z^{2N}}{(2N)!}\Big(1+\frac{z^2}{2N+1}\frac{b^2}{m^2}[-m^2(x-x')^2]^{1/2}\Big) \\
& \times\Big\{(x-x')^2b^2 - \frac{(x-x')_\alpha(x-x')_\beta}{(x-x')^2}(x-x')^2b^\alpha b^\beta\Big\}^N \\
& \times\sum_{n=0}^\infty\frac{z^{2n}}{n!}\Big(\frac{b^2}{m^2}\Big)^n[-m^2(x-x')^2]^{(n+1)/2}\,K_{n-1}\Big([-m^2(x-x')^2]^{1/2}\Big)\Big]_{x'\to x}\,. \qquad (35)
\end{aligned}
$$

---

[4] This approximation matches with the conventional Feynman diagram calculation of the *one-loop* vacuum polarization tensor.

Using the following formulas

$$
\begin{aligned}
&\lim_{x'\to x} \Phi_0(x,x') = 1 \ , \\
&\lim_{x'\to x} [-m^2(x-x')^2]^{1/2}\, K_{-1}\Big([-m^2(x-x')^2]^{1/2}\Big) = 1 \ , \\
&\lim_{x'\to x} [-m^2(x-x')^2]^{(n+1)/2}\, K_{n-1}\Big([-m^2(x-x')^2]^{1/2}\Big) = 0 \ , \qquad n = 1,2,3,\cdots \\
&\lim_{x'\to x} \frac{(x-x')_\mu (x-x')_\nu}{(x-x')^2} = c g_{\mu\nu} \ , \quad (c \text{ being an arbitrary contant})
\end{aligned} \tag{36}
$$

it is not difficult to see that each higher order term separately vanishes. Thus, we finally arrive at

$$
\langle J_\mu(x)\rangle = -\frac{ce^2}{2\pi^2}\tilde{F}_{\mu\nu} b^\nu \int_0^1 dz \ ,
$$

and equivalently

$$
\Gamma_{\rm CS} = \frac{ce^2}{4\pi^2}\int d^4x\, \epsilon^{\mu\nu\lambda\rho} b_\mu A_\nu F_{\lambda\rho} \int_0^1 dz \ ,
$$

which means that the all-order-in-b calculation does not alter the linear-in-b result and thus the functiom $f(b^2/m^2)$ in Eq. (5) vanishes. This complete our calculation.

We understand [6,7] the identity of the lowest-order calculation with the all-order calculation by following an argument of Coleman and Glashow [20]. Since in the expansion of the b-dependent vacuum polarization amplitude in powers of b (b-perturbation theory), all except the first order are free of linear divergences. Hence there is no ambiguity in evaluating higher order graphs. Momentarily let each of the two photons carry different momenta, say p_1 and p_2 (this means that the chiral inserions of $\not{b}\gamma_5$ carry non-zero momentum); Applying the Ward identity to this vacuum polarization amplitude, one finds that the amplitude is $O(p_1)$ and $O(p_2)$ at the same time, i.e., it is $O(p_1 p_2)$; now go to equal momenta, $p_1 = p_2 = p$, and observe that the amplitude must be $O(p^2)$. The Chern-Simons term one is seeking is $O(p)$; hence all these higher-order graphs do not contribute.

In summary, we have used the the Fock-Schwinger proper time method to calculate the induced Chern-Simons term arising from the Lorentz- and CPT-violating sector of quantum electrodynamics with a $b_\mu\bar{\psi}\gamma^\mu\gamma_5\psi$ term. Although there is an ambiguity in the calculation as mentioned in footnote 2, our all-order-in-b result coincides, *independently of this ambiguity*, with the linear-in-b result.

ACKNOWLEDGMENTS

This work was supported by a Korea Research Foundation grant (KRF-2000-015-DP0066).

REFERENCES

[1] S. Carroll, G. Field and R. Jackiw, Phys. Rev. D **41**, 1231 (1990); See also R. Jackiw, Comments on Modern Physics **A1**, 1 (1999).

[2] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (NY) **140**, 372 (1982); (E) **185**, 406 (1988).

[3] D. Colladay and V.A. Kostelecký, Phys. Rev. D **58**, 116002 (1998).

[4] J.-M. Chung and P. Oh, Phys. Rev D **60**, 067702 (1999).

[5] J.-M. Chung, Phys. Rev. D **60**, 127901 (1999).

[6] J.-M. Chung, Phys. Lett. **B461**, 138 (1999).

[7] M. Pérez-Victoria, Phys. Rev. Lett. **83**, 2518 (1999).

[8] R. Jackiw and V.A. Kostelecký, Phys. Rev. Lett. **82**, 3572 (1999); R. Jackiw, Int. J. Mod. Phys. **B14**, 2011 (2000).

[9] W. F. Chen, Phys. Rev D **60**, 085007 (1999).

[10] L.-H. Chan, hep-ph/9907349.

[11] M. B. Hott and J. L. Tomazelli, hep-th/9912251.

[12] M. Chaichian, W. F. Chen, and R. González Felipe, hep-th/0010129 v2.

[13] J. R. S. Nascimento, R. F. Ribeiro, and N. F. Svaiter, hep-th/0012039.

[14] A. P. Baeta Scarpelli, M. Sampaio, M. C. Nemes, hep-th/0010285.

[15] G. Bonneau, Nucl. Phys. **B593** 398 (2001).

[16] D. Colladay and V.A. Kostelecký, Phys. Rev. D **55**, 6760 (1997).

[17] L.-H. Chan, Phys. Rev. Lett. **57**, 1199 (1986); M. K. Gaillard, Nucl. Phys. **B268**, 669 (1986).

[18] V. Fock, Physik. Z. Sowjetunion **12**, 404 (1937).

[19] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[20] S. Coleman and S. Glashow, Phys. Rev. D **59**, 116008 (1999).